



NE 112 *Linear algebra for nanotechnology engineering*

1.6.10 Three inequalities

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Outline

- In this topic, we will
 - Describe the idea of monotonic increasing functions
 - Use this to prove three inequalities about complex numbers





The absolute value of real number

- For real numbers, you already are aware that

Theorem

For any real number x , $x \leq |x|$.

Proof:

If $x \geq 0$, by definition $x = |x|$.

Now, $0 \leq |x|$, and if $x < 0$, $x < 0 \leq |x|$, so $x < |x|$.

Therefore, $x = |x|$ or $x < |x|$.

Consequently, $x \leq |x|$. ■





Monotonic increasing functions

- Now, a real-valued function f of a real variable is *monotonic increasing* on an interval $[a, b]$ if

$$f(x_1) \leq f(x_2) \text{ whenever } a \leq x_1 \leq x_2 \leq b$$

Theorem

A differentiable real-valued function f of a real variable is monotonic increasing on an interval $[a, b]$ if and only if

$$f^{(1)}(x) \geq 0 \text{ for all } a < x < b.$$

Proof:

A consequence of the mean-value theorem from calculus. ■





Monotonic increasing functions

- Examples of monotonic increasing functions include:

Function	Interval
x^2	$[0, \infty)$
$\sqrt{x}$	$[0, \infty)$
$\ln(x)$	$(0, \infty)$
x^3	$\mathbf{R}$
e^x	$\mathbf{R}$

Domain of monotonicity





Monotonic increasing functions

- If

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n,$$

and all points are in a domain of monotonicity of f ,

$$f(x_1) \leq f(x_2) \leq f(x_3) \leq \cdots \leq f(x_n).$$

- In this course, we will use it as follows:
 - If $0 \leq x \leq y$, then $0 \leq x^2 \leq y^2$
 - If $0 \leq x \leq y$, then $0 \leq \sqrt{x} \leq \sqrt{y}$





Rectangular to polar

Theorem

Given a complex value z , $\Re(z) \leq |\Re(z)| \leq |z|$.

Proof:

Because $\Re(z)$ is a real value, $\Re(z) \leq |\Re(z)|$ is a consequence of the previous theorem.

If $z = \alpha + \beta j$, then because $\beta^2 \geq 0$, it follows that

$$\alpha^2 \leq \alpha^2 + \beta^2$$

By monotonicity,

$$\sqrt{\alpha^2} \leq \sqrt{\alpha^2 + \beta^2}$$

and by definition, this means $|\Re(z)| \leq |z|$. ■





Rectangular to polar

Theorem

Given a complex value z , $\Im m(z) \leq |\Im m(z)| \leq |z|$.

Proof:

Left to the viewer.





Triangle inequality for real numbers

- Suppose I told you that $|x| \leq 10$ and $|y| \leq 10$, but $|x + y| = 1368$?
- What if $|x| \leq 10$ and $|y| \leq 10$, but $|x + y| = 20.1$?
- In a sense, you already understand that:

Theorem

Given x and y real, then $|x + y| \leq |x| + |y|$.

Proof:

$$\begin{aligned}(x + y)^2 &= x^2 + 2xy + y^2 \\ &\leq |x|^2 + |2xy| + |y|^2 \\ &= |x|^2 + 2|x||y| + |y|^2 \\ &= (|x| + |y|)^2\end{aligned}$$

But $x^2 = |x|^2$ and $2xy \leq |2xy|$

But $|2xy| = |2||x||y| = 2|x||y|$

$$\text{Therefore } \sqrt{(x + y)^2} \leq \sqrt{(|x| + |y|)^2}$$

But $|x| + |y| \geq 0$

Thus, $|x + y| \leq |x| + |y|$. ■





Triangle inequality for complex numbers

- The same holds for complex numbers:

Theorem

Given w and z complex, then $|w + z| \leq |w| + |z|$.



Triangle inequality for complex numbers

Proof:

$$\begin{aligned} |w + z|^2 &= (w + z)(w + z)^* \\ &= (w + z)(w^* + z^*) \\ &= ww^* + wz^* + zw^* + zz^* \\ &= |w|^2 + 2\Re(wz^*) + |z|^2 \\ &\leq |w|^2 + 2|wz^*| + |z|^2 \\ &= |w|^2 + 2|w||z^*| + |z|^2 \\ &= |w|^2 + 2|w||z| + |z|^2 \\ &= (|w| + |z|)^2 \end{aligned}$$

$$\begin{aligned} (wz^*)^* &= w^* z^{**} \\ &= w^* z \\ &= zw^* \end{aligned}$$

Therefore, $\sqrt{|w + z|^2} \leq \sqrt{(|w| + |z|)^2}$

Thus, $|w + z| \leq |w| + |z|$. ■





Summary

- In this topic, you now
 - Understand the concept of monotonic increasing functions
 - Know the three inequalities:

$$\Re(z) \leq |\Re(z)| \leq |z|$$

$$\Im(z) \leq |\Im(z)| \leq |z|$$

$$|w + z| \leq |w| + |z|$$



References

- [1] https://en.wikipedia.org/wiki/Triangle_inequality



Acknowledgments

None so far.



Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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